

HOCHSCHILD AND CYCLIC HOMOLOGY OF CENTRAL EXTENSIONS OF PREPROJECTIVE ALGEBRAS OF ADE QUIVERS

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1. INTRODUCTION

Preprojective algebras of quivers were introduced by Gelfand and Ponomarev in 1979 in order to provide a model for quiver representations (in the special case of finite Dynkin quivers). Since then, preprojective algebras have found many other important applications, see e.g. [CBH]. Ironically, it is exactly in the case of finite Dynkin quivers, originally considered by Gelfand and Ponomarev, that preprojective algebras fail to have certain good properties enjoyed by the preprojective algebras of other connected quivers; for instance, their deformed versions are not flat. Motivated by this, the paper [ER] introduces central extensions of preprojective algebras of finite Dynkin quivers, and shows that they have better properties, in particular their deformed versions are flat. The following paper [ELR] computes the center Z and the trace space $A/[A, A]$ for the deformed preprojective algebra A ; the answer turns out to be related to the structure of the maximal nilpotent subalgebra of the simple Lie algebra attached to the quiver.

The goal of this paper is to generalize the results of [ELR] by calculating the additive structure of the Hochschild homology and cohomology of A and the cyclic homology of A , and to describe the universal deformation of A . Namely, we show that the (co)homology is periodic with period 4, and compute the first four (co)homology groups in each case. We plan to study the product structures on the (co)homology in a separate publication.

We note that the Hochschild cohomology of usual preprojective algebras A_0 (without central extension), together with the cup product, is studied in [ES] (in the case of type A). This is done by using the periodic resolution of A_0 with period 6 constructed by Schofield. This leads to the cohomology being periodic with period 6. We note that our periodic resolution with period 4 for A is quite similar to the Schofield resolution.¹ In a separate paper, we will apply our methods to computing the Hochschild (co)homology and cyclic homology of preprojective algebras of type D and E.

The structure of the paper is as follows. In Section 2 we discuss preliminaries. In Section 3, we define the periodic resolution with period 4 for A ,

¹The reason our resolution has smaller period than Schofield's is that A , unlike A_0 , has a symmetric invariant pairing, and hence the Nakayama automorphism of A , unlike that of A_0 , is the identity.

and use it to compute the Hochschild homology and cohomology of A . In Section 4, we compute the cyclic homology of A , and find the Hilbert series for all the homology and cohomology, using the results of [ELR] and combinatorial identities from [RS]. Finally, in Section 5 we use the result about $HH^2(A)$ to find a universal deformation of A (the deformation theory of A turns out to be unobstructed).

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2. PRELIMINARIES

2.1. Quivers, path algebras and preprojective algebras. Let Q be a quiver of ADE type with vertex set I and $|I| = r$. We define Q^* to be the quiver obtained from Q by reversing all of its arrows. We call $\bar{Q} = Q \cup Q^*$ the *double* of Q .

The concatenation of these arrows generate the *nontrivial paths* inside the quiver \bar{Q} . We define $e_i, i \in I$ to be the *trivial path* which starts and ends at i . The *path algebra* $P = \mathbb{C}\bar{Q}$ of Q over \mathbb{C} is the \mathbb{C} -algebra with basis the paths in \bar{Q} and the product xy of two paths x and y to be their concatenation if they are compatible and 0 if not. We define the *Lie bracket* $[x, y] = xy - yx$. By taking the quotient $P/(\sum_{a \in Q} [a, a^*])$, we obtain the *preprojective algebra*

denoted by Π_Q .

Let $R = \bigoplus_{i \in I} \mathbb{C}e_i$. Then P (and therefore Π_Q) is naturally an R -bimodule.

2.2. The symmetric bilinear form, roots and weights. We write $a \in Q$ to say that a is an arrow in Q . Let $h(a)$ denote its *head* and $t(a)$ its *tail*, i.e. for $a : i \rightarrow j$, $h(a) = j$ and $t(a) = i$. The *Ringel form* of Q is the bilinear form on \mathbb{Z}^I defined by

$$\langle \alpha, \beta \rangle = \sum_{i \in I} \alpha_i \beta_i - \sum_{a \in Q} \alpha_{t(a)} \beta_{h(a)}$$

for $\alpha, \beta \in \mathbb{Z}^I$. We define the *quadratic form* $q(\alpha) = \langle \alpha, \alpha \rangle$ and the *symmetric bilinear form* $(\alpha, \beta) = \langle \alpha, \beta \rangle + \langle \beta, \alpha \rangle$. It can be shown that q is positive definite for a finite Dynkin quiver Q .

We define the *set of roots* $\Delta = \{\alpha \in \mathbb{Z}^I | q(\alpha) = 1\}$.

We call the elements of \mathbb{C}^I *weights*. A weight $\mu = (\mu_i)$ is called *regular* if the inner product $(\mu, \alpha) \neq 0$ for all $\alpha \in \Delta$. We call the coordinate vectors $\varepsilon_i \in \mathbb{C}^I$ the *fundamental weights* and define ρ to be the sum of all fundamental weights.

We call $h = \frac{|\Delta|}{|I|}$ the *Coxeter number* of Q .

2.3. The centrally extended preprojective algebra. Let $\mu = (\mu_i)$ be a regular weight. We define the *centrally extended preprojective algebra* $A = A^\mu$ to be the quotient of $P[z]$ (z is a central variable) by the relation

$\sum_{a \in Q} [a, a^*] = z(\sum_{i \in I} \mu_i e_i)$. By taking the quotient $A/(z)$, we obtain the usual preprojective algebra $\Pi_Q = P/(\sum_{a \in Q} [a, a^*])$.

The grading on A is given by $\deg(R) = 0$, $\deg(a) = \deg(a^*) = 1$ and $\deg(z) = 2$.

From now on, we assume μ to be a generic weight or $\mu = \rho$.

2.4. The Hilbert series.

Definition 2.4.1. (The Hilbert series of vector spaces)

Let $W = \bigoplus_{d \geq 0} W(d)$ be a \mathbb{Z}_+ -graded vector space, with finite dimensional homogeneous subspaces. We define the Hilbert series $h_W(t)$ to be the series

$$h_W(t) = \sum_{d=0}^{\infty} \dim W[d] t^d.$$

Definition 2.4.2. (The Hilbert series of bimodules)

Let $W = \bigoplus_{d \geq 0} W(d)$ be a \mathbb{Z}_+ -graded bimodule over the ring $R = \bigoplus_{i \in I} \mathbb{C}$ (I is a finite set), so we can write $W = \bigoplus_{i,j \in I} W_{i,j}$. We define the Hilbert series $H_W(t)$ to be a matrix valued series with the entries

$$H_W(t)_{i,j} = \sum_{d=0}^{\infty} \dim W(d)_{i,j} t^d.$$

3. HOCHSCHILD HOMOLOGY/COHOMOLOGY AND CYCLIC HOMOLOGY OF A

3.1. Periodic projective resolution of A . Let V be the R -bimodule which is generated by the arrows in \bar{Q} (i.e. the degree 1-part of A). For a \mathbb{Z} -graded R -bimodule M , we denote $M[i]$ to be the bimodule M , shifted by degree i (i.e. $M(d) = M[i](d+i)$).

We want to compute Hochschild homology and cohomology of A , so we want to find a projective resolution of A .

Let

$$\begin{aligned} C_{-1} &= A, \\ C_0 &= A \otimes_R A, \\ C_1 &= (A \otimes_R V \otimes_R A) \oplus (A \otimes_R A)[2], \\ C_2 &= (A \otimes_R V \otimes_R A)[2] \oplus (A \otimes_R A)[2], \\ C_3 &= A \otimes_R A[4], \\ C_4 &= C_0[2h]. \end{aligned}$$

We define the following A -bimodule-homomorphisms $d_i : C_i \rightarrow C_{i-1}$:

$$d_0(b_1 \otimes b_2) = b_1 b_2,$$

$$d_1(b_1 \otimes \alpha \otimes b_2, b_3 \otimes b_4) = b_1 \alpha \otimes b_2 - b_1 \otimes \alpha b_2 + b_3 z \otimes b_4 - b_3 \otimes z b_4,$$

$$d_2(b_1 \otimes \alpha \otimes b_2, b_3 \otimes b_4) = (-b_1 z \otimes \alpha \otimes b_2 + b_1 \otimes \alpha \otimes z b_2 + \sum_{a \in \bar{Q}} \epsilon_a b_3 a \otimes a^* \otimes b_4 \\ + \sum_{a \in \bar{Q}} \epsilon_a b_3 \otimes a \otimes a^* b_4, -b_3 \mu \otimes b_4 + b_1 \alpha \otimes b_2 - b_1 \otimes \alpha b_2),$$

where we introduce the notation $\epsilon_a = \begin{cases} +1 & a \in Q \\ -1 & a \in Q^* \end{cases}$,

$$d_3(b_1 \otimes b_2) = (\sum_{a \in \bar{Q}} \epsilon_a b_1 a \otimes a^* \otimes b_2 + \sum_{a \in \bar{Q}} \epsilon_a b_1 \otimes a \otimes a^* b_2, b_1 z \otimes b_2 - b_1 \otimes z b_2),$$

$$d_4(b_1 \otimes b_2) = \sum b_1 x_i \otimes x_i^* b_2,$$

where $\{x_i\}$ is a basis of A and $\{x_i^*\}$ the dual basis under the (symmetric and nondegenerate) trace form $(x, y) = \text{Tr}(xy)$ introduced in [ELR, Section 2.2]. It is easy to see that d_4 is independent of the choice of the basis $\{x_i\}$. It is clear that all d_i are degree-preserving.

Using the trace form, it is easy to show that $\sum a x_i \otimes x_i^* = \sum x_i \otimes x_i^* a$ for any $a \in A$:

$$\sum a x_i \otimes x_i^* = \sum \sum (a x_i, x_j^*) x_j \otimes x_i^* = \sum \sum x_i \otimes (x_i^* a, x_j) x_j^* = \sum x_i \otimes x_i^* a.$$

This implies

$$d_4(b_1 \otimes b_2) = b_0(b_1 \otimes b_2) \sum x_i \otimes x_i^*$$

Theorem 3.1.1. *From the maps d_i we obtain the following projective resolution C_\bullet of A with period 4:*

$$\dots \xrightarrow{d_3[2h]} C_2[2h] \xrightarrow{d_2[2h]} C_1[2h] \xrightarrow{d_1[2h]} C_0[2h] \xrightarrow{d_4} C_3 \xrightarrow{d_3} C_2 \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0 \xrightarrow{d_0} A \rightarrow 0.$$

Proof. Let us first show that these C_i, d_i define a complex. We show that $d_i d_{i+1} = 0$ for $i \leq 3$ and $d_4 d_1[2h] = 0$:

$$d_0 d_1(b_1 \otimes \alpha \otimes b_2, b_3 \otimes b_4) = d_0(b_1 \alpha \otimes b_2 - b_1 \otimes \alpha b_2 + b_3 z \otimes b_4 - b_3 \otimes z b_4) = 0,$$

$$d_1 d_2(b_1 \otimes \alpha \otimes b_2, b_3 \otimes b_4) =$$

$$= d_1(-b_1 z \otimes \alpha \otimes b_2 + b_1 \otimes \alpha \otimes z b_2 + \sum_{a \in \bar{Q}} \epsilon_a b_3 a \otimes a^* \otimes b_4 \\ + \sum_{a \in \bar{Q}} \epsilon_a b_3 \otimes a \otimes a^* b_4, -b_3 \mu \otimes b_4 + b_1 \alpha \otimes b_2 - b_1 \otimes \alpha b_2) \\ = -b_1 z \alpha \otimes b_2 + b_1 z \otimes \alpha b_2 + b_1 \alpha \otimes z b_2 - b_1 \otimes \alpha z b_2 + \sum_{a \in \bar{Q}} \epsilon_a b_3 a a^* \otimes b_4 \\ - \sum_{a \in \bar{Q}} \epsilon_a b_3 a \otimes a^* b_4 + \sum_{a \in \bar{Q}} \epsilon_a b_3 a \otimes a^* b_4 - \sum_{a \in \bar{Q}} \epsilon_a b_3 \otimes a a^* b_4 - b_3 z \mu \otimes b_4 \\ + b_3 \mu \otimes z b_4 + b_1 \alpha z \otimes b_2 - b_1 \alpha \otimes z b_2 - b_1 z \otimes \alpha b_2 + b_1 \otimes z \alpha b_2 = 0$$

(since $\sum_{a \in \bar{Q}} \epsilon_a aa^* = z\mu$),

$$\begin{aligned}
d_2 d_3(b_1 \otimes b_2) &= \\
&= d_2 \left(\sum_{a \in \bar{Q}} \epsilon_a b_1 a \otimes a^* \otimes b_2 + \sum_{a \in \bar{Q}} \epsilon_a b_1 \otimes a \otimes a^* b_2, b_1 z \otimes b_2 - b_1 \otimes z b_2 \right) = \\
&= \left(- \sum_{a \in \bar{Q}} \epsilon_a b_1 a z \otimes a^* \otimes b_2 + \sum_{a \in \bar{Q}} \epsilon_a b_1 a \otimes a^* \otimes z b_2 - \sum_{a \in \bar{Q}} \epsilon_a b_1 z \otimes a \otimes a^* b_2 \right. \\
&\quad + \sum_{a \in \bar{Q}} \epsilon_a b_1 \otimes a \otimes z a^* b_2 + \sum_{a \in \bar{Q}} \epsilon_a b_1 z a \otimes a^* \otimes b_2 + \sum_{a \in \bar{Q}} \epsilon_a b_1 z \otimes a \otimes a^* b_2 \\
&\quad - \sum_{a \in \bar{Q}} \epsilon_a b_1 a \otimes a^* \otimes z b_2 - \sum_{a \in \bar{Q}} \epsilon_a b_1 \otimes a \otimes a^* z b_2, \\
&\quad - b_1 z \mu \otimes b_2 + b_1 \otimes z \mu b_2 + \sum_{a \in \bar{Q}} \epsilon_a b_1 a a^* \otimes b_2 - \sum_{a \in \bar{Q}} \epsilon_a b_1 a \otimes a^* b_2 \\
&\quad \left. + \sum_{a \in \bar{Q}} \epsilon_a b_1 a \otimes a^* b_2 - \sum_{a \in \bar{Q}} \epsilon_a b_1 \otimes a a^* b_2 \right) = 0,
\end{aligned}$$

$$\begin{aligned}
d_3 d_4(b_1 \otimes b_2) &= d_3 \left(\sum b_1 x_i \otimes x_i^* b_2 \right) = \\
&= \left(\sum_{a \in \bar{Q}} \sum \epsilon_a b_1 x_i a \otimes a^* \otimes x_i^* b_2 + \sum_{a \in \bar{Q}} \sum \epsilon_a b_1 x_i \otimes a \otimes a^* x_i^* b_2, \right. \\
&\quad \left. \sum b_1 x_i z \otimes x_i^* b_2 - \sum b_1 x_i \otimes z x_i^* b_2 \right).
\end{aligned}$$

Using the trace form, it is easy to show that $\sum x_i a \otimes x_i^* = \sum x_i \otimes a x_i^*$ for any $a \in A$:

$$\sum x_i a \otimes x_i^* = \sum \sum (x_i a, x_j^*) x_j \otimes x_i^* = \sum \sum x_i \otimes (a x_i^*, x_j) x_j^* = \sum x_i \otimes a x_i^*.$$

It follows that $\sum b_1 x_i z \otimes x_i^* b_2 - \sum b_1 x_i \otimes z x_i^* b_2 = 0$.

Similarly, $\sum x_i a \otimes b \otimes x_i^* = \sum x_i \otimes b \otimes a x_i^*$ for any $a \in A$. Therefore

$$\sum \epsilon_a b_1 x_i a \otimes a^* \otimes x_i^* b_2 = \sum \epsilon_a b_1 x_i \otimes a^* \otimes a x_i^* b_2 = \sum \underbrace{\epsilon_a^*}_{=-\epsilon_a} b_1 x_i \otimes a \otimes a^* x_i^* b_2,$$

so $d_3 d_4 = 0$.

$$\begin{aligned}
d_4 d_1[2h](b_1 \otimes \alpha \otimes b_2, b_3 \otimes b_4) &= \\
&= d_0(b_1 \alpha \otimes b_2 - b_1 \otimes \alpha b_2 + b_3 z \otimes b_4 - b_3 \otimes z b_4) \sum x_i \otimes x_i^* = 0.
\end{aligned}$$

Now we show exactness. Since the complex is periodic, it is enough to show exactness for C_0 , C_1 , C_2 and C_3 .

We recall the definition of Anick's resolution [An]. Denote $T_R W$ to be the tensor algebra of a graded R -bimodule W , $T_R^+ W$ its augmentation ideal. Let $L \subset T_R^+ W$ be an R -graded bimodule and $B = T_R W / (L)$. Then we have the following resolution:

$$(3.1.2) \quad B \otimes_R L \otimes_R B \xrightarrow{\partial} B \otimes_R W \otimes_R B \xrightarrow{f} B \otimes_R B \xrightarrow{m} B \rightarrow 0,$$

where m is the multiplication map, f is given by

$$f(b_1 \otimes w \otimes b_2) = b_1 w \otimes b_2 - b_1 \otimes w b_2$$

and ∂ is given by

$$\partial(b_1 \otimes l \otimes b_2) = b_1 \cdot D(l) \cdot b_2,$$

$$D : T_R^+ W \rightarrow B \otimes_R W \otimes_R B,$$

$$w_1 \otimes \dots \otimes w_n \mapsto \sum_{p=1}^n (\overline{w_1 \otimes \dots \otimes w_{p-1}}) \otimes w_p \otimes (\overline{w_{p+1} \otimes \dots \otimes w_n}),$$

where bar stands for the image in B of the projection map.

In our setting, $W = V \oplus Rz$, L the R -bimodule generated by $\sum_{a \in \bar{Q}} \epsilon_a a a^* - \mu z$

and $\alpha z - z\alpha \forall \alpha \in \bar{Q}$. Then $B = A$.

In Anick's resolution, $m = d_0$, $A \otimes_R W \otimes_R A$ can be identified with C_1 (via $A \otimes_R A[2] = A \otimes_R Rz \otimes_R A$), so that f becomes d_1 . Then $\text{Im}(\partial) = \text{Im}(d_2) \subset C_1$. This implies exactness in C_0 and C_1 .

For exactness in 2^{nd} and 3^{rd} term, we show that the complex

$$C_4 = C_0[2h] \xrightarrow{d_4} C_3 \xrightarrow{d_3} C_2 \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0 \xrightarrow{d_0} A = C_{-1} \rightarrow 0$$

is selfdual:

By replacing $C_4 = C_0[2h]$ by $\bar{C}_4 = \text{Im}(d_4)$, we get the complex

$$0 \rightarrow \bar{C}_4 \xrightarrow{\bar{d}_4} \bar{C}_3 \xrightarrow{\bar{d}_3} \bar{C}_2 \xrightarrow{\bar{d}_2} \bar{C}_1 \xrightarrow{\bar{d}_1} \bar{C}_0 \xrightarrow{\bar{d}_0} A \rightarrow 0.$$

Now, the map $\sum b_1 x_i \otimes x_i^* b_2 \mapsto b_1 b_2$ allows us to identify $\text{Im}(d_4) \cong A[2h]$ as A -bimodules so \bar{d}_4 becomes multiplication with $\sum x_i \otimes x_i^*$.

We introduce the following nondegenerate, bilinear forms:

On $A \otimes_R A$, let

$$(x \otimes y, a \otimes b) = \text{Tr}(xb) \text{Tr}(ya),$$

and on $A \otimes_R V \otimes_R A$, we define

$$(x \otimes \alpha \otimes y, a \otimes \beta \otimes b) = \text{Tr}(xb) \text{Tr}(ya)(\alpha, \beta),$$

where we define the form on V by

$$(\alpha, \beta) = \epsilon_\beta \delta_{\alpha^* \beta}$$

$$(\alpha, \beta \in \bar{Q} \text{ and } \delta_{ab} = \begin{cases} 1 & a = b \\ 0 & a \neq b \end{cases}).$$

Via the trace form $(x, y) = \text{Tr}(xy)$, we can identify $A \cong A^*$, $x \mapsto (x, -)$, and similarly we can use the forms from above to identify $A \otimes_R A \cong (A \otimes_R A)^*$ and $A \otimes_R V \otimes_R A \cong (A \otimes_R V \otimes_R A)^*$, which induces an identification $\bar{C}_i = \bar{C}_{3-i}^*$.

We claim the following: $\bar{d}_0^* = \bar{d}_4$, $\bar{d}_1^* = -\bar{d}_3$ and $\bar{d}_2^* \iota = \bar{d}_2$,
where $\iota(x, y) = (-x, y)$:

$$\begin{aligned} (\bar{d}_4(x), (b_1 \otimes b_2)) &= (\sum x x_i \otimes x_i^*, b_1 \otimes b_2) = \sum \text{Tr}(x x_i b_2) \text{Tr}(x_i^* b_1) \\ &= \sum (b_2 x, x_i)(x_i^*, b_1) = (b_2 x, b_1) = (x, b_1 b_2) \\ &= (x, \bar{d}_0(b_1 \otimes b_2)). \end{aligned}$$

For $\alpha, \beta \in \bar{Q}$,

$$\begin{aligned} (-\bar{d}_3(x \otimes y), (b_1 \otimes \alpha \otimes b_2, b_3 \otimes b_4)) &= \\ &= ((-\sum_{a \in \bar{Q}} \epsilon_a x a \otimes a^* \otimes y - \sum_{a \in \bar{Q}} \epsilon_a x \otimes a \otimes a^* y, -x z \otimes y + x \otimes z y), \\ &\quad (b_1 \otimes \alpha \otimes b_2, b_3 \otimes b_4)) \\ &= -\text{Tr}(x \alpha b_2) \text{Tr}(y b_1) + \text{Tr}(x b_2) \text{Tr}(\alpha y b_1) - \text{Tr}(x z b_4) \text{Tr}(y b_3) \\ &\quad + \text{Tr}(x b_4) \text{Tr}(z y b_3) \\ &= \text{Tr}(x b_2) \text{Tr}(y b_1 \alpha) - \text{Tr}(x \alpha b_2) \text{Tr}(y b_1) + \text{Tr}(x b_4) \text{Tr}(y b_3 z) \\ &\quad - \text{Tr}(x z b_4) \text{Tr}(y b_3) \\ &= (x \otimes y, b_1 \alpha \otimes b_2 - b_1 \otimes \alpha b_2 + b_3 z \otimes b_4 - b_3 \otimes z b_4) \\ &= (x \otimes y, d_1(b_1 \otimes \alpha \otimes b_2, b_3 \otimes b_4)), \end{aligned}$$

$$\begin{aligned} (\bar{d}_2(b_1 \otimes \alpha \otimes b_2, b_3 \otimes b_4), (c_1 \otimes \beta \otimes c_2, c_3 \otimes c_4)) &= \\ &= ((-b_1 z \otimes \alpha \otimes b_2 + b_1 \otimes \alpha \otimes z b_2 + \sum_{a \in \bar{Q}} \epsilon_a b_3 a \otimes a^* \otimes b_4 \\ &\quad + \sum_{a \in \bar{Q}} \epsilon_a b_3 \otimes a \otimes a^* b_4, -b_3 \mu \otimes b_4 + b_1 \alpha \otimes b_2 - b_1 \otimes \alpha b_2), \\ &\quad (c_1 \otimes \beta \otimes c_2, c_3 \otimes c_4)) \\ &= -\text{Tr}(b_1 z c_2) \text{Tr}(b_2 c_1)(\alpha, \beta) + \text{Tr}(b_1 c_2) \text{Tr}(z b_2 c_1)(\alpha, \beta) \\ &\quad + \text{Tr}(b_3 \beta c_2) \text{Tr}(b_4 c_1) - \text{Tr}(b_3 c_2) \text{Tr}(\beta b_4 c_1) \\ &\quad - \text{Tr}(b_3 \mu c_4) \text{Tr}(b_4 c_3) + \text{Tr}(b_1 \alpha c_4) \text{Tr}(b_2 c_3) - \text{Tr}(b_1 c_4) \text{Tr}(\alpha b_2 c_3) \\ &= \text{Tr}(b_1 c_2) \text{Tr}(b_2 c_1 z)(\alpha, \beta) - \text{Tr}(b_1 z c_2) \text{Tr}(b_2 c_1)(\alpha, \beta) \\ &\quad - \text{Tr}(b_1 c_4) \text{Tr}(b_2 c_3 \alpha) + \text{Tr}(b_1 \alpha c_4) \text{Tr}(b_2 c_3) \\ &\quad - \text{Tr}(b_3 \mu c_4) \text{Tr}(b_4 c_3) - \text{Tr}(b_3 c_2) \text{Tr}(b_4 c_1 \beta) + \text{Tr}(b_3 \beta c_2) \text{Tr}(b_4 c_1) \\ &= ((b_1 \otimes \alpha \otimes b_2, b_3 \otimes b_4), (c_1 z \otimes \beta \otimes c_2 - c_1 \otimes \beta \otimes z c_2 \\ &\quad + \sum_{a \in \bar{Q}} \epsilon_a c_3 a \otimes a^* \otimes c_4 + \sum_{a \in \bar{Q}} \epsilon_a c_3 \otimes a \otimes a^* c_4, \\ &\quad -c_3 \otimes \mu c_4 - c_1 \beta \otimes c_2 + c_1 \otimes \beta c_2)) \\ &= ((b_1 \otimes \alpha \otimes b_2, b_3 \otimes b_4), \bar{d}_2(-c_1 \otimes \beta \otimes c_2, c_3 \otimes c_4)). \end{aligned}$$

Now, the selfduality of our complex \bar{C}_\bullet and exactness in \bar{C}_0 and \bar{C}_1 implies exactness in \bar{C}_2 and \bar{C}_3 . \square

3.2. Computation of Hochschild cohomology/homology. Now we use the projective resolution C_\bullet to compute the Hochschild cohomology and homology groups of A . Let us write $A^e = A \otimes_R A^{op}$.

Theorem 3.2.1. *The Hochschild cohomology groups of A are:*

$$\begin{aligned} HH^0(A) &= Z \text{ (the center of } A), \\ HH^{4n+1}(A) &= (Z \cap \mu^{-1}[A, A])[-2nh - 2], \\ HH^{4n+2}(A) &= A/([A, A] + \mu Z)[-2nh - 2], \\ HH^{4n+3}(A) &= A_+/[A, A][-2nh - 4], \\ HH^{4n+4}(A) &= Z/A_{top}[-2(n+1)h] \end{aligned}$$

where $n \geq 0$, and A_{top} is the top-degree part of A .

Proof. Apply the functor $Hom_{A^e}(-, A)$ on C_\bullet , identify

$$Hom_{A^e}(A \otimes_R A, A) \cong A^R$$

($\phi \in Hom_{A^e}(A \otimes_R A, A)$ is determined by $\phi(1 \otimes 1) = a \in A$ and observe $ra = \phi(r \otimes 1) = \phi(1 \otimes r) = ar, \forall r \in R$. We write $a \circ -$ for ϕ) and

$$Hom_{A^e}(A \otimes_R V \otimes_R A, A) \cong (A \otimes_R V)^R[-2]$$

($\sum_{a \in \bar{Q}} x_a \otimes a^*$ is identified with the homomorphism ψ which maps each element

$1 \otimes a \otimes 1$ to x_a ($a \in \bar{Q}$), we write $\sum_{a \in \bar{Q}} x_a \otimes a^*$ for $\psi(-)$)

to obtain the Hochschild cohomology complex

$$\cdots \leftarrow A^R[-2h] \xleftarrow{d_4^*} A^R[-4] \xleftarrow{d_3^*} \begin{array}{c} (A \otimes_R V)^R[-4] \\ \oplus \\ A^R[-2] \end{array} \xleftarrow{d_2^*} \begin{array}{c} (A \otimes_R V)^R[-2] \\ \oplus \\ A^R[-2] \end{array} \xleftarrow{d_1^*} A^R \leftarrow 0.$$

$$\begin{aligned} d_1^*(x)(b_1 \otimes \alpha \otimes b_2, b_3 \otimes b_4) &= x \circ d_1(b_1 \otimes \alpha \otimes b_2, b_3 \otimes b_4) = \\ &= b_1 \alpha x b_2 - b_1 x \alpha b_2 + b_3 z x b_4 - b_3 x z b_4 = b_1[\alpha, x]b_2, \end{aligned}$$

so

$$d_1^*(x) = \left(\sum_{a \in \bar{Q}} [a, x] \otimes a^*, 0 \right).$$

Let $\alpha = \sum_{a \in \bar{Q}} r_a a$, $r_a \in R$.

$$\begin{aligned}
d_2^*(\sum_{a \in \bar{Q}} x_a \otimes a^*, 0)(b_1 \alpha \otimes b_2, b_3 \otimes b_4) &= (\sum_{a \in \bar{Q}} x_a \otimes a^*) \circ d_2(b_1 \alpha \otimes b_2, b_3 \otimes b_4) = \\
&= \sum_{a \in \bar{Q}} (x_a \otimes a^*) \circ (-b_1 z \otimes \alpha \otimes b_2 + b_1 \otimes \alpha \otimes z b_2 + \sum_{\beta \in \bar{Q}} \epsilon_\beta b_3 \beta \otimes \beta^* \otimes b_4 \\
&\quad + \sum_{\beta \in \bar{Q}} \epsilon_\beta b_3 \otimes \beta \otimes \beta^* b_4) = \\
&= \sum_{a \in \bar{Q}} (-b_1 z r_a x_a b_2 + b_1 r_a x_a z b_2) - \sum_{a \in \bar{Q}} \epsilon_a b_3 a^* x_a b_4 + \sum_{a \in \bar{Q}} \epsilon_a b_3 x_a a^* b_4 \\
&= \sum_{a \in \bar{Q}} \epsilon_a b_3 [x_a, a^*] b_4,
\end{aligned}$$

so

$$d_2^*(\sum_{a \in \bar{Q}} x_a \otimes a^*, 0) = (0, \sum_{a \in \bar{Q}} \epsilon_a [x_a, a^*]).$$

$$\begin{aligned}
d_2^*(0, y)(b_1 \otimes \alpha \otimes b_2, b_3 \otimes b_4) &= y \circ d_2(b_1 \otimes \alpha \otimes b_2, b_3 \otimes b_4) = \\
&= y \circ (-b_3 \mu \otimes b_4 + b_1 \alpha \otimes b_2 - b_1 \otimes \alpha b_2) = -b_3 \mu y b_4 + b_1 \alpha y b_2 - b_1 y \alpha b_2,
\end{aligned}$$

so

$$d_2^*(0, y) = (-\sum_{a \in \bar{Q}} [y, a] \otimes a^*, -\mu y).$$

Putting this together, we obtain:

$$d_2^*(\sum_{a \in \bar{Q}} x_a \otimes a^*, y) = (-\sum_{a \in \bar{Q}} [y, a] \otimes a^*, -\mu y + \sum_{a \in \bar{Q}} \epsilon_a [x_a, a^*]).$$

$$\begin{aligned}
d_3^*(\sum_{a \in \bar{Q}} x_a \otimes a^*, 0)(b_1 \otimes b_2) &= (\sum_{a \in \bar{Q}} x_a \otimes a^*) \circ d_3(b_1 \otimes b_2) = \\
&= (\sum_{a \in \bar{Q}} x_a \otimes a^*) \circ (\sum_{\alpha \in \bar{Q}} \epsilon_\alpha b_1 \alpha \otimes \alpha^* \otimes b_2 + \sum_{\alpha \in \bar{Q}} \epsilon_\alpha b_1 \otimes \alpha \otimes \alpha^* b_2, \\
&\quad b_1 z \otimes b_2 - b_1 \otimes z b_2) = \\
&= \sum_{a \in \bar{Q}} (-\epsilon_a b_1 a^* x_a b_2 + \epsilon_a b_1 x_a a^* b_2) = \sum_{a \in \bar{Q}} \epsilon_a b_1 [x_a, a^*] b_2, \\
d_3^*(0, y)(b_1 \otimes b_2) &= \\
&= y \circ (\sum_{\alpha \in \bar{Q}} \epsilon_\alpha b_1 \alpha \otimes \alpha^* \otimes b_2 + \sum_{\alpha \in \bar{Q}} \epsilon_\alpha b_1 \otimes \alpha \otimes \alpha^* b_2, b_1 z \otimes b_2 - b_1 \otimes z b_2) \\
&= b_1 z y b_2 - b_1 y z b_2 = 0,
\end{aligned}$$

so we get

$$d_3^*(\sum_{a \in \bar{Q}} x_a \otimes a^*, y) = \sum_{a \in \bar{Q}} \epsilon_a [x_a, a^*].$$

$$d_4^*(x)(b_1 \otimes b_2) = x \circ (\sum b_1 x_i \otimes x_i^* b_2) = \sum b_1 x_i x x_i^* b_2,$$

so

$$d_4^*(x) = \sum x_i x x_i^*.$$

Now, we want to compute the Hochschild cohomology (since the complex is periodic, $HH^i(A) = HH^{i+4}(A)[2h] \forall i \geq 1$, so it is enough to do the calculations until HH^4):

$HH^0(A) = Z$ (the center of A), since a cocycle $x \in \ker d_1^*$ lies in A^R and has to satisfy $\sum_{a \in \bar{Q}} [a, x] \otimes a^* = 0$, i.e. commute with all $a \in \bar{Q}$.

$HH^1(A) = (Z \cap \mu^{-1}[A, A])[-2] : (\sum_{a \in \bar{Q}} x_a \otimes a^*, y)$ is a cocycle if $\sum_{a \in \bar{Q}} [y, a] \otimes a^* = 0$ (i.e. $y \in Z$) and $y = \mu^{-1} \sum_{a \in \bar{Q}} \epsilon_a [x_a, a^*]$ (since μ is invertible) which implies $y \in \mu^{-1}[A, A]$. Since $\sum_{a \in \bar{Q}} \epsilon_a [a^*, x_a] = 0$ implies that $x_a = [a, x]$

(we refer to [ELR, Corollary 3.5.] where this statement follows from the exactness of the complex in the 1st term) for some $x \in A$, and $\sum_{a \in \bar{Q}} [a, x] \otimes a^*$

lies in $\text{Im} d_1^*$, $HH^1(A)$ is controlled only by $y \in (Z \cap \mu^{-1}[A, A])[-2]$. Since $[A(1), A] = [A, A]$, any $y \in (Z \cap \mu^{-1}[A, A])[-2]$ also gives rise to a cocycle.

$HH^2(A) = A/([A, A] + \mu Z)[-2] : \text{An element } (\sum_{a \in \bar{Q}} x_a \otimes a^*, y) \text{ is a cocycle if } \sum_{a \in \bar{Q}} \epsilon_a [x_a, a^*] = 0$, so $x_a = [x, a]$ for some $x \in A^R$, (where x is unique up to a central element), so cocycles are of the form $(\sum_{a \in \bar{Q}} [x, a] \otimes a^*, y)$. The coboundaries are spanned by $(\sum_{a \in \bar{Q}} [x, a] \otimes a^*, \mu x)$ (where the first component determines x uniquely modulo Z) and $(0, \sum_{a \in \bar{Q}} \epsilon_a [x_a, a^*])$ (where the image is $[A, A]^R$). It follows that

$$HH^2(A) = A^R/([A, A]^R + \mu Z)[-2] = A/([A, A] + \mu Z)[-2].$$

$HH^3(A) = A_+/[A, A][-4] : \text{We denote } A_+ \text{ to be the positive degree part of } A. d_4^*(x) = \sum x_i x x_i^* \text{ is zero if } x \text{ has positive degree (since } x_i x x_i^* \text{ exceeds the top degree } 2h - 4).$

Observe also that d_4^* injects R into A_{top} . Since $A = \oplus e_k A e_j$, we can choose a basis $\{x_i\}$, such that these elements all belong to a certain subspace $e_k A e_j$ for some k, j . We denote $\{x_{i'}^{j,k}\}$ the subbasis of $\{x_i\}$ which spans $e_k A e_j$.

Assume that $0 = d_4^*(\sum_{j=1}^r \lambda_j e_j)$. Then $\forall k$,

$$\begin{aligned} 0 &= \sum_{j=1}^r \lambda_j \text{Tr}(\sum_i e_k x_i e_j x_i^*) = \sum_{j=1}^r \lambda_j \sum_{i', j, k} \underbrace{(x_{i'}^{j, k}, (x_{i'}^{j, k})^*)}_{=1} = \sum_{j=1}^r \lambda_j \dim e_k A e_j \\ &= \sum_j \lambda_j \sum_d \dim e_k A[d] e_j = H_A(1)_{k, j} = \sum_j \lambda_j \left(\frac{h}{2-C} \right)_{k, j}. \end{aligned}$$

The last equality follows from [ER, Theorem 3.2.]. Since the matrix $\frac{h}{2-C}$ is nondegenerate, all $\lambda_j = 0$.

So we see that the images $d_4^*(e_j)$ are nonzero and linearly independent. So the cocycles are the elements in A_+^R , and the coboundaries are $\sum_{a \in \bar{Q}} \epsilon_a [x_a, a^*]$

which generate $[A, A]^R$. Therefore $HH^3(A) = A_+^R/[A, A]^R[-4] = A_+/[A, A](-4)$.

$HH^4(A) = Z/A_{top}[-2h]$: Since $d_5^* = d_1^*$, the cocycles are the central elements. From the above discussion about the image of d_4^* and the fact that A_{top} is r -dimensional, it follows that the coboundaries are the top degree elements of A . \square

Similarly, we compute the Hochschild homology groups of A .

Theorem 3.2.2. *The Hochschild homology groups of A are:*

$$\begin{aligned} HH_0(A) &= A/[A, A], \\ HH_{4n+1}(A) &= A/([A, A] + \mu Z)[2nh + 2], \\ HH_{4n+2}(A) &= (Z \cap \mu^{-1}[A, A])[2nh + 2], \\ HH_{4n+3}(A) &= Z/A_{top}[2nh + 4], \\ HH_{4n+4}(A) &= A_+/[A, A][2(n+1)h]. \end{aligned}$$

Proof. Apply the functor $(A \otimes_{A^e} -)$ to C_\bullet , identify

$$A \otimes_{A^e} (A \otimes_R A) \cong A^R$$

$(a \otimes (b \otimes c) = cab \otimes 1 \otimes 1 \mapsto cab$ and observe

$\forall a \in A, r \in R : ar = a \otimes (r \otimes 1) = a \otimes (1 \otimes r) = ra$ and

$$A \otimes_{A^e} (A \otimes_R V \otimes_R A) \cong (A \otimes_R V)^R$$

(via $a \otimes (b \otimes \alpha \otimes c) = cab \otimes (1 \otimes \alpha \otimes 1) \mapsto cab \otimes \alpha$).

We get the following periodic complex for computing the Hochschild homology:

$$\cdots \rightarrow A^R[2h] \xrightarrow{d'_4} A^R[4] \xrightarrow{d'_3} \begin{array}{c} (A \otimes_R V)^R[2] \\ \oplus \\ A^R[2] \end{array} \xrightarrow{d'_2} \begin{array}{c} (A \otimes_R V)^R \\ \oplus \\ A^R[2] \end{array} \xrightarrow{d'_1} A^R \rightarrow 0.$$

The differentials become:

$$\begin{aligned} d'_1\left(\sum_{a \in \bar{Q}} x_a \otimes a, y\right) &= 1 \otimes d_1\left(\sum_{a \in \bar{Q}} x_a \otimes a \otimes 1, y \otimes 1\right) \\ &= 1 \otimes \left(\sum_{a \in \bar{Q}} x_a a \otimes 1 - \sum_{a \in \bar{Q}} x_a \otimes a + yz \otimes 1 - y \otimes z\right) = \sum_{a \in \bar{Q}} [x_a, a], \end{aligned}$$

$$\begin{aligned} d'_2\left(\sum_{a \in \bar{Q}} x_a \otimes a, y\right) &= 1 \otimes d_2\left(\sum_{a \in \bar{Q}} x_a \otimes a \otimes 1, y \otimes 1\right) = \\ &= 1 \otimes \left(-\sum_{a \in \bar{Q}} x_a z \otimes a \otimes 1 + \sum_{a \in \bar{Q}} x_a \otimes a \otimes z + \sum_{\alpha \in \bar{Q}} \epsilon_\alpha y \alpha \otimes \alpha^* \otimes 1 \right. \\ &\quad \left. + \sum_{\alpha \in \bar{Q}} \epsilon_\alpha y \otimes \alpha \otimes \alpha^*, -y\mu \otimes 1 + \sum_{a \in \bar{Q}} x_a a \otimes 1 - \sum_{a \in \bar{Q}} x_a \otimes a\right) \\ &= \left(-\sum_{a \in \bar{Q}} x_a z \otimes a + \sum_{a \in \bar{Q}} z x_a \otimes a + \sum_{\alpha \in \bar{Q}} \epsilon_\alpha y \alpha \otimes \alpha^* + \sum_{\alpha \in \bar{Q}} \epsilon_\alpha \alpha^* y \otimes \alpha, \right. \\ &\quad \left.-y\mu + \sum_{a \in \bar{Q}} x_a a - \sum_{a \in \bar{Q}} a x_a\right) = \left(\sum_{\alpha \in \bar{Q}} \epsilon_\alpha [y, \alpha] \otimes \alpha^*, \sum_{a \in \bar{Q}} [x_a, a] - y\mu\right), \end{aligned}$$

$$\begin{aligned} d'_3(x) &= 1 \otimes d_3(x \otimes 1) = \\ &= (1 \otimes \left(\sum_{a \in \bar{Q}} \epsilon_a x a \otimes a^* \otimes 1 + \sum_{a \in \bar{Q}} \epsilon_a x \otimes a \otimes a^*\right), 1 \otimes (xz \otimes 1 - x \otimes z)) \\ &= \left(\sum_{a \in \bar{Q}} \epsilon_a x a \otimes a^* + \sum_{a \in \bar{Q}} \epsilon_a a^* x \otimes a, xz - zx\right) = \left(\sum_{a \in \bar{Q}} \epsilon_a [x, a] \otimes a^*, 0\right), \end{aligned}$$

$$d'_4(x) = 1 \otimes d_4(x \otimes 1) = 1 \otimes \sum x x_i \otimes x_i^* = \sum x_i^* x x_i.$$

Now, we compute the homology (and since the complex is periodic, $HH_i(A) = HH_{i+4}(A)$ for $i > 0$, so it is enough to calculate the homology up to HH_4):

$HH_0(A) = A/[A, A]$: the boundaries are of the form $\sum_{a \in \bar{Q}} [x_a, a]$, and they generate $[A, A]^R$. So $HH_0(A) = A^R/[A, A]^R = A/[A, A]$ follows.

$HH_1(A) = A/([A, A] + \mu Z)[2]$: The cycle condition $\sum_{a \in \bar{Q}} [x_a, a] = 0$ implies $x_a = \epsilon_a [x, a^*]$ for some $x \in A$ (again, we refer to the result $H_1 = 0$ in [ELR, Corollary 3.5.]), so the cycles are $(\sum_{a \in \bar{Q}} \epsilon_a [x, a^*] \otimes a, y)$.

The boundaries are of the form $(\sum_{a \in \bar{Q}} \epsilon_a [x, a^*] \otimes a, \sum_{\alpha \in \bar{Q}} [x_\alpha, \alpha] + \mu x)$ (where the first component determines x uniquely modulo Z). So

$$HH_1(A) = A^R/([A, A]^R + \mu Z)[2] = A/([A, A] + \mu Z)[2].$$

$HH_2(A) = Z \cap \mu^{-1}[A, A][2]$: The cycle conditions are $\sum_{\alpha \in \bar{Q}} \epsilon_\alpha[y, \alpha] \otimes \alpha^* = 0$ (this tells us $y \in Z$) and $\sum_{a \in \bar{Q}} [x_a, a] - y\mu = 0$, so $y \in \mu^{-1}[A, A]$ and x_a unique up to an element of the form $\epsilon_a[x, a^*]$ for some $x \in A$. So the cycles are of the form $(\sum_{a \in \bar{Q}} x_a \otimes a, y)$, $y \in Z \cap \mu^{-1}[A, A]$, x_a uniquely controlled by y (mod $\epsilon_a[x, a^*]$), and the boundaries have the form $(\sum_{a \in \bar{Q}} \epsilon_a[x, a^*] \otimes a, 0)$, i.e. homology is controlled only by y now. So $HH_2(A) = Z \cap \mu^{-1}[A, A][2]$.
 $HH_3(A) = Z/A_{top}[4]$: The cycle condition $\sum_{a \in \bar{Q}} \epsilon_a[x, a] \otimes a^* = 0$ implies that the cycles are the central elements Z . The boundaries $\sum x_i^* x x_i$ consist of the top degree part of A , so $HH_3(A) = Z/A_{top}[4]$.
 $HH_4(A) = A_+/[A, A][2h]$: $\ker d_4 = A_+^R$, $\text{Im } d_5 = \text{Im } d_1 = [A, A]^R$, therefore $HH_4(A) = A_+^R/[A, A]^R = A_+/[A, A]$. \square

3.3. The intersection $Z \cap \mu^{-1}[A, A]$. We found $Z \cap \mu^{-1}[A, A]$ as the $(4i + 2)$ -th homology and $(4i + 1)$ -th cohomology group, so to understand the (co)homology of A better, we are interested in its structure.

Now, we define the following Hilbert series:

$$\begin{aligned} q(t) &= h_{Z \cap \mu^{-1}[A, A]}(t), \\ q_*(t) &= h_{A/([A, A] + \mu Z)}(t). \end{aligned}$$

To relate both to each other, we prove the following

Proposition 3.3.1. *The trace form defines a nondegenerate pairing $(Z \cap \mu^{-1}[A, A]) \times A/([A, A] + \mu Z) \rightarrow \mathbb{C}$.*

Proof. Since the trace form is nondegenerate on A , it is enough to show that $(Z \cap \mu^{-1}[A, A])^\perp \subset [A, A] + \mu Z$, or equivalently $([A, A] + \mu Z)^\perp \subset Z \cap \mu^{-1}[A, A]$. The latter follows from $[A, A]^\perp \subset Z$, since

$$(x, [y_1, y_2]) = \text{Tr}(x[y_1, y_2]) = \text{Tr}([x, y_1]y_2) = ([x, y_1], y_2) = 0 \forall y_1, y_2 \in A$$

implies $[x, y_1] = 0$, i.e. $x \in Z$. \square

Corollary 3.3.2. *$q(t)$ and $q_*(t)$ are palindromes of each other, i.e. $q(t) = t^{2h-4}q_*(1/t)$.*

Let us define the Hilbert series $p(t) = h_{A/\mu^{-1}[A, A]}(t)$. We recall from [ELR, end of section 2.2.] that $p(t) = \sum_{i=1}^r (1 + t^2 + \dots + t^{2(m_i-1)})$ where the m_i are the exponents of the root system. Since the trace form also defines a nondegenerate pairing $Z \times A/[A, A] \rightarrow \mathbb{C}$ (see [ELR, Corollary 2.2.]), it follows for the Hilbert series $p_*(t) = h_Z(t)$ that $p(t) = t^{2h-4}p_*(1/t)$. Since $zZ \subset \mu^{-1}[A, A]$ is spanned by even degree elements, we see that Z is generated as a $\mathbb{C}[z]$ -module by elements of degree $2(m_i - 1)$.

Proposition 3.3.3. *We have*

$$q_*(t) \geq p(t) - \sum_{i=1}^r t^{2(m_i-1)} = \sum_{i=1}^r \left(1 + t^2 + \dots + t^{2(m_i-2)}\right).$$

Proof. From the exact sequence

$$0 \rightarrow Z/(Z \cap \mu^{-1}[A, A]) \rightarrow A/\mu^{-1}[A, A] \rightarrow A/(\mu^{-1}[A, A] + Z) \rightarrow 0$$

we obtain the equation

$$q_*(t) = p(t) - h_{Z/(Z \cap \mu^{-1}[A, A])}(t).$$

Since $zZ \subset \mu^{-1}[A, A]$ ($z = \mu^{-1} \sum_{a \in \bar{Q}} [a, a^*] \in \mu^{-1}[A, A]$), we have the inequality

$$h_{Z/(Z \cap \mu^{-1}[A, A])}(t) \leq h_{Z/zZ}(t) = \sum_{i=1}^r t^{2(m_i-1)},$$

and our inequality

$$q_*(t) \geq p(t) - \sum_{i=1}^r t^{2(m_i-1)}$$

follows. □

Theorem 3.3.4. *The inequality from above is an equality:*

$$q_*(t) = p(t) - \sum_{i=1}^r t^{2(m_i-1)}.$$

We will prove this in the next section where we compute the cyclic homology groups of A . From this, we get a result for our intersection space:

Corollary 3.3.5. $Z \cap \mu^{-1}[A, A] = zZ$.

3.4. Cyclic homology of A . The Connes differentials B_i (see [Lo, 2.1.7.]) give us an exact sequence

$$R \xrightarrow{B_{-1}} HH_0(A) \xrightarrow{B_0} HH_1(A) \xrightarrow{B_1} HH_2(A) \xrightarrow{B_2} HH_3(A) \xrightarrow{B_3} HH_4(A) \xrightarrow{B_4} \dots$$

In our case, we have the following exact sequence:

$$\begin{aligned} R &\xrightarrow{B_{-1}} A/[A, A] \xrightarrow{B_0} A/([A, A] + \mu Z)[2] \xrightarrow{B_1} Z \cap \mu^{-1}[A, A][2] \xrightarrow{B_2} Z/A_{top}[4] \\ &\xrightarrow{B_3} A_+/[A, A][2h] \xrightarrow{B_4} \dots, \end{aligned}$$

and the B_i are all degree-preserving.

We define the *reduced cyclic homology* (see [Lo, 2.2.13.])

$$\begin{aligned} \overline{HC}_i(A) &= \ker(B_{i+1} : HH_{i+1}(A) \rightarrow HH_{i+2}(A)) \\ &= \text{Im}(B_i : HH_i(A) \rightarrow HH_{i+1}(A)). \end{aligned}$$

Theorem 3.4.1. *We get the following cyclic homology groups:*

$$\begin{aligned}\overline{HC}_{4n}(A) &= A_+/[A, A][2nh], \\ \overline{HC}_{4n+1}(A) &= 0, \\ \overline{HC}_{4n+2}(A) &= Z/A_{top}[2nh+4], \\ \overline{HC}_{4n+3}(A) &= 0.\end{aligned}$$

Proof. First we observe that $B_{4n+3} = 0$, since the elements of $Z/A_{top}[4]$ have degree $\leq (2h-6)+4 = 2h-2$ and the elements in $A_+/[A, A][2h]$ have degree $\geq 2h+1$. So we have for each n the exact sequences

$$\begin{aligned}0 \rightarrow \frac{A_+}{[A, A][2nh]} \xrightarrow{B_{4n}} \frac{A}{[A, A] + \mu Z}[2nh+2] \xrightarrow{B_{4n+1}} (Z \cap \mu^{-1}[A, A])[2nh+2] \\ \xrightarrow{B_{4n+2}} \frac{Z}{A_{top}}[2nh+4] \rightarrow 0.\end{aligned}$$

The only thing to show is that $W := \overline{HC}_{4n+1}(A) = \text{Im} B_{4n+1} = 0$. We will use the following theorem from [EG]:

Theorem 3.4.2. *Let $\chi_{\overline{HC}(A)}(t) = \sum a_k t^k$, the Euler characteristic of $\overline{HC}(A)$. Then*

$$\prod_{k=1}^{\infty} (1-t^k)^{-a_k} = \prod_{s=1}^{\infty} \det H_A(t^s) = \prod_{s=1}^{\infty} \left(\frac{1-t^{2hs}}{1-t^{2s}} \right)^r \frac{1}{\det(1-Ct^s+t^{2s})},$$

where C is the adjacency matrix of the quiver Q .

Since

$$\chi_{\overline{HC}(A)}(t) = \frac{1}{1-t^{2h}} (h_{A_+/[A, A]}(t) - h_W(t) + h_{Z/A_{top}}(t)t^4),$$

to show $W = 0$, it is enough to show that if we set

$$\frac{1}{1-t^{2h}} (h_{A_+/[A, A]}(t) + h_{Z/A_{top}}(t)t^4) = \sum b_k t^k,$$

then

$$\prod_{k=1}^{\infty} (1-t^k)^{b_k} = \prod_{s=1}^{\infty} \left(\frac{1-t^{2s}}{1-t^{2hs}} \right)^r \det(1-Ct^s+t^{2s}).$$

We have

$$\begin{aligned}h_{A_+/[A, A]} &= p(t) - r = \sum_{i=1}^r \frac{t^2 - t^{2m_i}}{1-t^2} \text{ and} \\ h_{Z/A_{top}} t^4 &= \sum_{i=1}^r \frac{t^{2(m_i-1)} - t^{2h-4}}{1-t^2} t^4 = \sum_{i=1}^r \frac{t^{2m_i+2} - t^{2h}}{1-t^2}.\end{aligned}$$

From these, we get that

$$\sum_{k=1}^{\infty} b_k t^k = (1+t^{2h}+t^{4h}+\dots) \sum_{i=1}^r (t^2+t^4+\dots+t^{2m_i-2}+0+t^{2m_i+2}+\dots+t^{2h-2}),$$

$$b_k = 0 \text{ if } k \text{ is odd}$$

$$b_{2k} = \begin{cases} 0 & \text{if } k \text{ is divisible by } h \\ r - \#\{i : m_i = p\} & \text{if } k \equiv p \pmod{h} \end{cases}.$$

$$\prod_{k=1}^{\infty} (1 - t^k)^{b_k} = \prod_{\substack{n \neq 0 \\ n \pmod{h}}} (1 - t^{2n})^r / \prod_{\substack{n \geq 0 \\ i \in I}} (1 - t^{2(m_i + nh)})$$

Now, it comes down to showing that

$$\prod_{s=1}^{\infty} \det(1 - Ct^s + t^{2s}) = \prod_{k=1}^{\infty} (1 - q^k)^{n_k},$$

where $q = t^2$ and $n_k = \begin{cases} 0 & \text{if } n \text{ is divisible by } h \\ -\#\{i : m_i = p\} & \text{if } n \equiv p \pmod{h} \end{cases}$
(recall that the m_i are the exponents of our root system), for the different Dynkin quivers of type A_{n-1} , D_{n+1} , E_6 , E_7 and E_8 . Here we will use the identities for $\det(1 - Ct + t^2) = \prod_{j=1}^r (t^2 - e^{2\pi i m_j / h})$ from [RS, Corollary 4.5].

Case 1: $Q = A_{n-1}$

The exponents are $1, \dots, n-1$ and the Coxeter number is $h = n$.

$$\det(1 - Ct + t^2) = \frac{1 - t^{2n}}{1 - t^2},$$

so if we set

$$\prod_{k=1}^{\infty} (1 - q^k)^{n_k} = \prod_{s=1}^{\infty} \det(1 - Ct^s + t^{2s}) = \prod_{s=1}^{\infty} \frac{1 - q^{ns}}{1 - q^s}$$

then

$$n_k = \begin{cases} 0 & \text{if } n|k \\ -1 & \text{if } n \nmid k \end{cases}.$$

Case 2: $Q = D_{n+1}$

The exponents are $1, 3, \dots, 2n-1, n$ and the Coxeter number is $h = 2n$.

$$\det(1 - Ct + t^2) = \frac{(1 - t^4)(1 - t^{4n})}{(1 - t^2)(1 - t^{2n})},$$

so

$$\prod_{k=1}^{\infty} (1 - q^k)^{n_k} = \prod_{s=1}^{\infty} \det(1 - Ct^s + t^{2s}) = \prod_{s=1}^{\infty} \frac{(1 - q^{2s})(1 - q^{2ns})}{(1 - q^s)(1 - q^{ns})}$$

implies that

$$n_k = \operatorname{div}(k, 2n) - \operatorname{div}(k, n) + \operatorname{div}(k, 2) - 1,$$

where we denote $\text{div}(p, q) = \begin{cases} 1 & \text{if } q|p \\ 0 & \text{if } q \nmid p \end{cases}$.

$$n_k = \begin{cases} 0 - 0 + 0 - 1 = -1 & k \text{ odd}, k \not\equiv 0, n \pmod{2n} \\ 0 - 0 + 1 - 1 = 0 & k \text{ even}, k \not\equiv 0, n \pmod{2n} \\ 0 - 1 + 1 - 1 = -1 & k \text{ even}, k \equiv n \pmod{2n} \\ 0 - 1 + 0 - 1 = -2 & k \text{ odd}, k \equiv n \pmod{2n} \\ 1 - 1 + 1 - 1 = 0 & k \equiv 0 \pmod{2n} \end{cases}.$$

Case 3: $Q = E_6$

The exponents are 1, 4, 5, 7, 8, 11 and the Coxeter number is $h = 12$.

$$\det(1 - Ct + t^2) = \frac{(1 - t^{24})(1 - t^4)(1 - t^6)}{(1 - t^{12})(1 - t^8)(1 - t^2)},$$

then

$$\prod_{k=1}^{\infty} (1 - q^k)^{n_k} = \prod_{s=1}^{\infty} \det(1 - Ct^s + t^{2s}) = \prod_{s=1}^{\infty} \frac{(1 - q^{12s})(1 - q^{2s})(1 - q^{3s})}{(1 - q^{6s})(1 - q^{4s})(1 - q^s)}$$

implies

$$n_k = \text{div}(k, 12) + \text{div}(k, 2) + \text{div}(k, 3) - \text{div}(k, 6) - \text{div}(k, 4) - 1.$$

Observe that if we have a prime factorization $q = a^2b$ (a, b distinct), then

$$\text{div}(k, q) + \text{div}(k, a) + \text{div}(k, b) - \text{div}(k, ab) - \text{div}(k, a^2) - 1$$

is -1 if k and q are relatively prime or if $k \equiv la^2 \pmod{q}$ ($l \neq 0$) and 0 else.

This proves our case for $12 = 2^2 \cdot 3$.

Case 4: $Q = E_7$

The exponents are 1, 5, 7, 9, 11, 13, 17 and the Coxeter number is $h = 18$.

$$\det(1 - Ct + t^2) = \frac{(1 - t^{36})(1 - t^6)(1 - t^3)}{(1 - t^{18})(1 - t^{12})(1 - t^2)},$$

so

$$\prod_{k=1}^{\infty} (1 - q^k)^{n_k} = \prod_{s=1}^{\infty} \det(1 - Ct^s + t^{2s}) = \prod_{s=1}^{\infty} \frac{(1 - q^{18s})(1 - q^{3s})(1 - q^{2s})}{(1 - q^{9s})(1 - q^{6s})(1 - q^s)}$$

implies

$$n_k = \text{div}(k, 18) + \text{div}(k, 3) + \text{div}(k, 2) - \text{div}(k, 9) - \text{div}(k, 6) - 1.$$

We use the same argument as above, for $18 = 2 \cdot 3^2$.

Case 5: $Q = E_8$

The exponents are 1, 7, 11, 13, 17, 19, 23, 29 and the Coxeter number is

$h = 30$.

$$\det(1 - Ct + t^2) = \frac{(1 - t^{60})(1 - t^{10})(1 - t^6)(1 - t^4)}{(1 - t^{30})(1 - t^{20})(1 - t^{12})(1 - t^2)},$$

then

$$\prod_{k=1}^{\infty} (1 - q^k)^{n_k} = \prod_{s=1}^{\infty} \det(1 - Ct^s + t^{2s}) = \prod_{s=1}^{\infty} \frac{(1 - q^{30s})(1 - q^{5s})(1 - q^{3s})(1 - q^{2s})}{(1 - q^{15s})(1 - q^{10s})(1 - q^{6s})(1 - q^s)}$$

implies

$$\begin{aligned} n_k &= \operatorname{div}(k, 30) + \operatorname{div}(k, 5) + \operatorname{div}(k, 3) + \operatorname{div}(k, 2) \\ &\quad - \operatorname{div}(k, 15) - \operatorname{div}(k, 10) - \operatorname{div}(k, 6) - 1. \end{aligned}$$

We use a similar argument here: If we have a prime factorization $q = abc$ (a, b, c distinct), then

$\operatorname{div}(k, q) + \operatorname{div}(k, a) + \operatorname{div}(k, b) + \operatorname{div}(k, c) - \operatorname{div}(k, ab) - \operatorname{div}(k, bc) - \operatorname{div}(k, ac) - 1$
is -1 if k and q are relatively prime and 0 else.

This proves our case for $30 = 2 \cdot 3 \cdot 5$. □

Proof. (of Theorem 3.3.4):
From the isomorphism

$$(Z \cap \mu^{-1}[A, A])[2] \xrightarrow{B_2} Z/A_{top}$$

we obtain the equation $t^2 q(t) = t^4 \sum_{i=1}^r (t^{2(m_i-1)} + \dots + t^{2h-6})$, so

$$q(t) = \sum_{i=1}^r (t^{2m_i} + \dots + t^{2h-4}).$$

Recall the duality of exponents, i.e. $m_{r+1-i} = h - m_i$. Then we get

$$\begin{aligned} q_*(t) &= t^{2h-4} q(1/t) = t^{2h-4} \sum_{i=1}^r (t^{-2m_i} + \dots + t^{-2h+4}) \\ &= t^{2h-4} \sum_{i=1}^r (t^{-2(h-m_i)} + \dots + t^{-2h+4}) \\ &= \sum_{i=1}^r (1 + \dots + t^{2(m_i-2)}) = p(t) - \sum_{i=1}^r t^{2(m_i-1)}. \end{aligned}$$

□

4. UNIVERSAL DEFORMATION OF A

Definition 4.0.3. For any weight $\lambda = (\lambda_i)$, we define the algebra

$$A(\lambda) = P[z]/\left(\sum_{a \in \bar{Q}} [a, a^*] = z\mu + \sum_{i=1}^r \lambda_i e_i\right)$$

and introduce a deformation parametrized by formal variables c_i^j , $1 \leq i \leq r$, $1 \leq j \leq h-1$:

$$A(\lambda)_c = P[z][[c]]/\left(\sum_{a \in \bar{Q}} [a, a^*] = z\mu + \sum_{i=1}^r \lambda_i e_i + \sum_{i=1}^r \sum_{j=1}^{h-1} c_i^j z^j e_i\right).$$

Theorem 4.0.4. This deformation is flat $\forall \lambda \in R$, i.e. $A(\lambda)_c$ is free over $\mathbb{C}[[c]]$, and

$$A(\lambda)_c/(c) = A(\lambda).$$

Proof. It is sufficient to check flatness for generic λ . From [ER, end of section 3.2.], we know that for generic λ , $A(\lambda) = \oplus \text{End} V_\alpha$ is a semisimple algebra. So it suffices to show that the representation V_α can be deformed to a representation of $A(\lambda)_c$ for all λ .

We recall from [CBH, Theorem 4.3.] that $\forall \beta \in R$, such that $\beta \cdot \alpha = 0$, it exists an α -dimensional irreducible representation V_α of P , such that

$$\sum_{a \in \bar{Q}} [a, a^*] = \sum_{i=1}^r \beta_i e_i.$$

If we set $z = \gamma \in \mathbb{C}$ in $A(\lambda)_c$, then the relation becomes

$$\sum_{a \in \bar{Q}} [a, a^*] = \sum_{i=1}^r e_i (\lambda_i + \gamma(\mu_i + c_i^1) + \gamma^2 c_i^2 + \dots).$$

Then for $\alpha = \sum_{i=1}^r \alpha_i e_i$, since the trace of $[a, a^*]$ is zero, the condition to have an α -dimensional representation of $A(\lambda)_c$ (i.e. a representation of P satisfying the above relation) is

$$\sum_{i=1}^r \alpha_i (\lambda_i + \gamma(\mu_i + c_i^1) + \gamma^2 c_i^2 + \dots) = 0.$$

By Hensel's lemma, this equation in $\mathbb{C}[[c]]$ has a unique solution γ , such that its constant term $\gamma_0 \in \mathbb{C}$ satisfies $\sum_{i=1}^r \alpha_i (\lambda_i + \gamma_0) = 0 \Rightarrow \gamma_0 = -\frac{\sum \alpha_i \lambda_i}{\sum \alpha_i}$. \square

In particular, if we treat λ as formal parameter, then $A(\lambda)_c$ is a flat deformation of $A(0)$.

Let E be the linear span of $z^j e_i$, $0 \leq j \leq h-2$, $1 \leq i \leq r$. From [ELR, Proposition 2.4.] we know that the projection map $E \rightarrow A/[A, A]$ is

surjective. Then the deformation $A(\lambda)_c$ is parametrized by E which gives us a natural map $\eta : E \rightarrow HH^2(A)$. On the other hand, the isomorphism $HH^2(A) = A/([A, A] + \mu Z)$ in Theorem 3.2.1 induces a projection map $\theta : E \rightarrow HH^2(A)$.

Proposition 4.0.5. *The maps $\theta, \eta : E \rightarrow HH^2(A)$ are identical.*

Proof. We have the following commutative diagram which connects our periodic projective resolution with the bar resolution of A ,

$$\begin{array}{ccccccc}
 (A \otimes V \otimes A[2]) & & (A \otimes V \otimes A) & & A \otimes A & & A \\
 \oplus & \xrightarrow{d_2} & \oplus & \xrightarrow{d_1} & & \xrightarrow{d_0} & \\
 (A \otimes A[2]) & & (A \otimes A[2]) & & & & \\
 f_2 \downarrow & & \downarrow f_1 & & \parallel & & \parallel \\
 A^{\otimes 4} & \xrightarrow{\tilde{d}_2} & A^{\otimes 3} & \xrightarrow{\tilde{d}_1} & A^{\otimes 2} & \xrightarrow{\tilde{d}_0} & A,
 \end{array}$$

where we define

$$f_1(b_1 \otimes \alpha \otimes b_2, b_3 \otimes b_4) = b_1 \otimes \alpha \otimes b_2 + b_3 \otimes z \otimes b_4$$

and

$$f_2(b_1 \otimes \alpha \otimes b_2, b_3 \otimes b_4) = -b_1 \otimes z \otimes \alpha \otimes b_2 + b_1 \otimes \alpha \otimes z \otimes b_2 + \sum_{a \in Q} \epsilon_a b_3 \otimes a \otimes a^* \otimes b_4.$$

Let us check the commutativity of the diagram:

$$\begin{aligned}
 \tilde{d}_1 f_1(b_1 \otimes \alpha \otimes b_2, b_3 \otimes b_4) &= \\
 &= b_1 \alpha \otimes b_2 - b_1 \otimes \alpha b_2 + b_3 z \otimes b_4 - b_3 \otimes z b_4 = d_1(b_1 \otimes \alpha \otimes b_2, b_3 \otimes b_4),
 \end{aligned}$$

$$\begin{aligned}
 f_1 d_2(b_1 \otimes \alpha \otimes b_2, b_3 \otimes b_4) &= \\
 &= f_1(-b_1 z \otimes \alpha \otimes b_2 + b_1 \otimes \alpha \otimes z b_2 + \sum_{a \in \bar{Q}} \epsilon_a b_3 a \otimes a^* \otimes b_4 \\
 &\quad + \sum_{a \in \bar{Q}} \epsilon_a b_3 \otimes a \otimes a^* b_4, -b_3 \mu \otimes b_4 + b_1 \alpha \otimes b_2 - b_1 \otimes \alpha b_2) \\
 &= -b_1 z \otimes \alpha \otimes b_2 - b_1 \otimes z \otimes \alpha b_2 + b_1 \alpha \otimes z \otimes b_2 + b_1 \otimes \alpha \otimes z b_2 \\
 &\quad + \sum_{a \in \bar{Q}} \epsilon_a b_3 a \otimes a^* \otimes b_4 - b_3 \otimes z \mu \otimes b_4 + \sum_{a \in \bar{Q}} \epsilon_a b_3 \otimes a \otimes a^* b_4 \\
 &= \tilde{d}_2 f_2(b_1 \otimes \alpha \otimes b_2, b_3 \otimes b_4).
 \end{aligned}$$

We apply $\text{Hom}_{A^e}(-, A)$ to the above diagram:

$$\begin{array}{ccccc}
 (A \otimes V)^R[2] & \xleftarrow{d_2^*} & (A \otimes V)^R[2] & \xleftarrow{d_1^*} & A^R \\
 \oplus & & \oplus & & \\
 (A^R[2]) & & A^R & & \\
 f_2^* \uparrow & & \uparrow f_1^* & & \parallel \\
 \text{Hom}_{A^e}(A^{\otimes 4}, A) & \xleftarrow{(\tilde{d}_2)^*} & \text{Hom}_{A^e}(A^{\otimes 3}, A) & \xleftarrow{(\tilde{d}_1)^*} & A^R.
 \end{array}$$

The map f_2^* induces a natural isomorphism on $HH^2(A)$, so via this identification we want to prove that $f_2^*\eta = \theta$.

The element $\gamma := \sum \gamma_i^j z^j e_i$, $\gamma_i^j \in \mathbb{C}$ defines the 1-parameter deformation

$$A^\gamma = A[[\hbar]] / \sum_{a \in Q} [a, a^*] = z + \hbar \left(\sum_{i=1}^r \gamma_i^0 e_i + \sum_{i=1}^r \sum_{j=1}^{h-2} \gamma_i^j z^j e_i \right),$$

so the cocycle $\eta(\gamma)$ is defined to be a bilinear map m on $A \times A$ (where we identify

$$\text{Hom}_{A^e}(A^{\otimes 4}, A) = \text{Hom}_k(A \otimes A, A)$$

here), such that for $a, b \in A$,

$$a * b \equiv ab + \hbar m(a, b) \pmod{\hbar^2}$$

where $"*"$ is the product in A^γ . This gives us:

$$\begin{aligned}
 f_2^*\eta(\gamma)(b_1 \otimes \alpha \otimes b_2, b_3 \otimes b_4) &= \eta(\gamma)f_2(b_1 \otimes \alpha \otimes b_2, b_3 \otimes b_4) = \\
 &= \eta(\gamma)(-b_1 \otimes z \otimes \alpha \otimes b_2 + b_1 \otimes \alpha \otimes z \otimes b_2 + \sum_{a \in Q} \epsilon_a b_3 \otimes a \otimes a^* \otimes b_4) \\
 &= b_1(m(z, \alpha) - m(\alpha, z))b_2 + b_3 \left(\sum_{a \in Q} m(a, a^*) - m(a^*, a) \right) b_4 \\
 &= b_3 \left(\sum_{i=1}^r \gamma_i^0 e_i + \sum_{i=1}^r \sum_{j=1}^{h-2} \gamma_i^j z^j e_i \right) b_4 = \theta(\gamma)(b_1 \otimes \alpha \otimes b_2, b_3 \otimes b_4).
 \end{aligned}$$

We obtain the second to last equality by:

$$0 + \hbar(m(z, \alpha) - m(\alpha, z)) = z * \alpha - \alpha * z = 0 \text{ and}$$

$$\begin{aligned}
 z\mu + \hbar \left(\sum_{a \in Q} m(a, a^*) - m(a^*, a) \right) &= \sum_{a \in Q} (a * a^* - a^* * a) \\
 &= z\mu + \hbar \left(\sum_{i=1}^r \gamma_i^0 e_i + \sum_{i=1}^r \sum_{j=1}^{h-2} \gamma_i^j z^j e_i \right).
 \end{aligned}$$

This finishes our proof that $f_2^*\eta = \theta$. \square

We see that the map $E \rightarrow HH^2(A)$ induced by the deformation $A(\lambda)_c$ is just the projection map. From this we can derive the universal deformation of A very easily.

Let $E' \subset E$ be the subspace which is complimentary to $\ker(\theta : E \rightarrow A/([A, A] + \mu Z))$ with basis w_i, \dots, w_s , and choose formal parameters t_i, \dots, t_s . The subdeformation A' of A , parametrized by $E' \subset E$ is:

$$A' = P[z][[t_1, \dots, t_s]] / \left(\sum_{a \in \bar{Q}} [a, a^*] = \mu z + \sum_{i=1}^s t_i w_i \right).$$

Theorem 4.0.6. *A' is the universal deformation of A .*

Proof. $\eta : E' \rightarrow HH^2(A)$ is the map induced by the deformation A' . Since θ induces an isomorphism $E' \rightarrow A/([A, A] + \mu Z) = HH^2(A)$, by Proposition 4.0.5 η is an isomorphism and therefore induces a universal deformation. \square

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